Univariate Functions for the Hilbert-Schmidt Volumes of the Real and Complex Separable Two-Qubit Systems

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Abstract

The (complex) two-qubit systems comprise a 15-dimensional convex set and the real two-qubit systems, a 9-dimensional convex set. While formulas for the Hilbert-Schmidt volumes of these two sets are known — owing to recent important work of Sommers and Życzkowski (*J. Phys. A* **36**, 10115 [2003]) — formulas have not been so far obtained for the volumes of the *separable* subsets. We reduce these two problems to the determination of certain functions ($f_{real}(\mu)$ and $f_{complex}(\mu)$) of a *single* variable $\mu = \sqrt{\frac{\rho_{11}\rho_{44}}{\rho_{22}\rho_{33}}}$, where ρ is the corresponding 4×4 density matrix, and the ρ_{ii}'s ($i = 1, \ldots 4$) its diagonal entries. The desired separable volumes are, then, of the form $V_{sep/real} = 2\int_0^1 jac_{real}(\mu) f_{real}(\mu) d\mu$ and $V_{complex/real} = 2\int_0^1 jac_{complex}(\mu) f_{complex}(\mu) d\mu$. Here jac denotes the corresponding (known) jacobian function. We provide estimates of the two sets of $f(\mu)$'s and V's.

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I. INTRODUCTION

In a pair of major, skillful papers, making use of the theory of random matrices [1], Sommers and Życzkowski were able to derive explicit formulas for the volumes occupied by the $d = (n^2 - 1)$-dimensional convex set of $n \times n$ (complex) density matrices (as well as the $d = \frac{(n-1)(n+2)}{2}$-dimensional convex set of real (symmetric) $n \times n$ density matrices), both in terms of the Hilbert-Schmidt (HS) metric [2] — inducing the flat, Euclidean geometry — and the Bures metric [3] (cf. [4]). Of course, it would be of obvious considerable quantum-information-theoretic interest in the cases that n is a composite number, to also obtain HS and Bures volume formulas restricted to those states that are separable — the sum of product states — in terms of some factorization of n [5]. Then, by taking ratios — employing these Sommers-Życzkowski results — one would obtain corresponding separability *probabilities*. (In an *asymptotic* regime, in which the dimension of the state space grows to infinity, Aubrun and Szarek recently concluded [6] that for qubits and larger-dimensional particles, the proportion of the states that are separable is superexponentially small in the dimension of the set.)

In particular, again for the 15-dimensional complex case, $n = 4 = 2 \times 2$, numerical evidence has been adduced that the Bures volume of separable states is (quite elegantly) $2^{-15}(\frac{\sqrt{2}-1}{3}) \approx 4.2136 \cdot 10^{-6}$ [7, Table VI] and the HS volume $(5\sqrt{3})^{-7} \approx 2.73707 \cdot 10^{-7}$ [8, eq. (41)]. Then, taking ratios (using the corresponding Sommers-Życzkowski results), we have the derived conjectures that the Bures separability probability is $\frac{1680(\sqrt{2}-1)}{\pi^8} \approx 0.0733389$ and the HS one, considerably larger, $\frac{2^2 \cdot 3 \cdot 7^2 \cdot 11 \cdot 13\sqrt{3}}{5^4 \pi^6} \approx 0.242379$ [8, eq. (43), but misprinted as 5^3 not 5^4 there]. (Szarek, Bengtsson and Życzkowski — motivated by the numerical findings of [8, 9] — have recently formally demonstrated "that the probability to find a random state to be separable equals 2 times the probability to find a random boundary state to be separable, provided the random states are generated uniformly with respect to the Hilbert-Schmidt (Euclidean) distance. An analogous property holds for the set of positive-partial-transpose states for an arbitrary bipartite system" [10] (cf. [11]). These authors also noted [10, p. L125] that "one could try to obtain similar results for a general class of multipartite systems". In this latter vein, recent numerical analyses of ours give some [but certainly not yet conclusive] indication that for the *three*-qubit *tri*separable states, there is an analogous probability ratio of 6 — rather than 2.)

However, the analytical derivation of (conjecturally) exact formulas for these HS and Bures (as well as other, such as the Kubo-Mori [12] and Wigner-Yanase [8, 13]) separable volumes has seemed quite remote — the only analytic progress to report so far being certain exact formulas when the number of dimensions of the 15-dimensional space of 4×4 density matrices has been severely curtailed (nullifying or holding constant *most* of the 15 parameters) to $d \leq 3$ [14, 15] (cf. [16]). Most notably, in this research direction, in [15, Fig. 11], we were able to find a highly interesting/intricate (one-dimensional) continuum ($-\infty < \beta < \infty$) of two-dimensional (the associated parameters being b_1, the mean, and σ_q^2, the variance of the Bell-CHSH observable) HS separability probabilities, in which the *golden ratio* [17] was featured, among other items. (The associated HS volume element — $\frac{1}{32\beta(1+\beta)} d\beta db_q d\sigma_q^2$ — is independent of b_1 and σ_q^2 in this three-dimensional scenario.) Further, in [14], building upon work of Jakóbczyk and Siennicki [18], we obtained a remarkably wide-ranging variety of exact HS separability ($n = 4, 6$) and PPT (positive partial transpose) ($n = 8, 9, 10$) probabilities based on *two*-dimensional sections of sets of (generalized) Bloch vectors corresponding to $n \times n$ density matrices.

The full $d = 9$ and/or $d = 15$, $n = 4$ real and complex two-qubit scenarios are quite daunting — due to the numerous separability constraints at work, some being active [binding] in certain regions and in complementary regions, inactive [nonbinding]. "The geometry of the 15-dimensional set of separable states of two qubits is not easy to describe" [10, p. L125]. We will seek to make substantial progress in these directions here, by recasting both these problems within a *one*-dimensional framework.

To proceed in our study, we employ the (quite simple) form of parameterization of the density matrices put forth by Bloore [19, 20] some thirty years ago. (Of course, there are a number of other possible parametrizations [21, 22, 23, 24, 25, 26, 27], a number of which we have also utilized in various studies [28, 29] to estimate volumes of separable states. Our greatest progress at this stage, in terms of increasing dimensionality, has been achieved with the Bloore parameterization — due to a certain computationally attractive feature of it, allowing us to decouple diagonal and non-diagonal parameters — as detailed shortly below.)

II. BLOORE PARAMETERIZATION OF THE DENSITY MATRICES

The main presentation of Bloore [19] was made in terms of the 3×3 ($n = 3$) density matrices. It is clearly easily extendible to cases $n > 3$. The fundamental idea is to scale the off-diagonal elements ($\rho_{ij}, i \neq j$) of the density matrix in terms of the square roots of the diagonal entries (ρ_{ii}). That is, we set (introducing the new [Bloore] variables z_{ij}),

$$\rho_{ij} = \sqrt{\rho_{ii}\rho_{jj}} z_{ij}. \tag{1}$$

This allows the determinant of ρ (and analogously all its principal minors) to be expressible as the product ($|\rho| = AB$) of two factors, one ($A = \Pi_{i=1}^{4} \rho_{ii}$) of which is itself simply the product of (nonnegative) diagonal entries (ρ_{ii}). In the real $n = 4$ case under investigation here — we have

$$B = \left(z_{34}^2 - 1\right) z_{12}^2 + 2\left(z_{14}\left(z_{24} - z_{23}z_{34}\right) + z_{13}\left(z_{23} - z_{24}z_{34}\right)\right) z_{12} - z_{23}^2 - z_{24}^2 - z_{34}^2 + \tag{2}$$
$$z_{14}^2\left(z_{23}^2 - 1\right) + z_{13}^2\left(z_{24}^2 - 1\right) + 2z_{23}z_{24}z_{34} + 2z_{13}z_{14}\left(z_{34} - z_{23}z_{24}\right) + 1,$$

involving (only) the z_{ij}'s ($i > j$), where $z_{ji} = z_{ij}$ [19, eqs. (15), (17)]. Since, clearly, the factor A is positive in all nondegenerate cases ($\rho_{ii} > 0$), one can — by only analyzing B — essentially ignore the diagonal entries, and thus reduce by $(n - 1)$ the dimensionality of the problem of finding nonnegativity conditions to impose on ρ. This is the feature we will seek to maximally exploit here. A fully analogous decoupling property holds in the complex case.

It is, of course, necessary and sufficient for ρ to serve as a density matrix (that is, an Hermitian, nonnegative definite, trace one matrix) that all its principal minors be nonnegative [30]. The condition — quite natural in the Bloore parameterization — that all the principal 2×2 minors be nonnegative requires simply that $-1 \leq z_{ij} \leq 1, i \neq j$. The joint conditions that all the principal minors be nonnegative are not as readily apparent. But for the 9-dimensional *real* case $n = 4$ — that is, $\Im(\rho_{ij}) = 0$ — we have been able to obtain one such set, using the Mathematica implementation of the *cylindrical algorithm decomposition* [31]. (The set of solutions of any system of real algebraic equations and inequalities can be decomposed into a finite number of "cylindrical" parts [32].) Applying it, we were able to express the conditions that an arbitrary 9-dimensional 4×4 real density matrix ρ must fulfill. These took the form, $z_{12}, z_{13}, z_{14} \in [-1, 1]$ and

$$z_{23} \in [Z_{23}^-, Z_{23}^+], z_{24} \in [Z_{24}^-, Z_{24}^+], z_{34} \in [Z_{34}^-, Z_{34}^+], \tag{3}$$

where

$$Z_{23}^{\pm} = z_{12}z_{13} \pm \sqrt{1-z_{12}^2}\sqrt{1-z_{13}^2}, Z_{24}^{\pm} = z_{12}z_{14} \pm \sqrt{1-z_{12}^2}\sqrt{1-z_{14}^2}, \tag{4}$$
$$Z_{34}^{\pm} = \frac{z_{13}z_{14} - z_{12}z_{14}z_{23} - z_{12}z_{13}z_{24} + z_{23}z_{24} \pm s}{1-z_{12}^2},$$

and

$$s = \sqrt{-1 + z_{12}^2 + z_{13}^2 - 2z_{12}z_{13}z_{23} + z_{23}^2}\sqrt{-1 + z_{12}^2 + z_{14}^2 - 2z_{12}z_{14}z_{24} + z_{24}^2}. \tag{5}$$

Making use of these results, we were able to confirm *via* exact symbolic integrations, the (formally demonstrated) result of Życzkowski and Sommers [2] that the HS volume of the *real* two-qubit ($n = 4$) states is $\frac{\pi^4}{60480} \approx 0.0016106$. (This result was also achievable through a somewhat different Mathematica computation, using the implicit integration feature first introduced in version 5.1. That is, the only integration limits employed were that $z_{ij} \in [-1, 1], i \neq j$ — *broader* than those in (3) — while the Boolean constraints were imposed that the determinant of ρ and *one* [all that is needed to ensure nonnegativity] of its principal 3×3 minors be nonnegative.)

A. Determinant of the Partial Transpose

However, when we tried to combine these integration limits (3) with the (Peres-Horodecki [33, 34, 35] $n = 4$) separability constraint that the determinant ($C = |\rho_{PT}|$) of the partial transpose of ρ be nonnegative [36, Thm. 5], we exceeded the memory availabilities of our workstations. In general, the term C — unlike the earlier term B — unavoidably involves the diagonal entries (ρ_{ii}), so the dimension of the accompanying integration problems must increase, it would seem, we initially thought — in the 9-dimensional real $n = 4$ case from 6 to 9.

However, we then noted that, in fact, the dimensionality of the required integrations must only essentially be increased by one (rather than three), since C turns out to be (aside from the necessarily nonnegative factor of A) expressible solely in terms of the (six, in the real case) distinct z_{ij}'s and the square root ratio

$$\mu = \sqrt{\frac{\rho_{11}\rho_{44}}{\rho_{22}\rho_{33}}}. \tag{6}$$

(Considering μ as fixed, this is the equation of an hyperboloid of one sheet [37, p. 227].) That is,

$$C \equiv |\rho_{PT}| = A\Big(-z_{14}^2\mu^4 + 2z_{14}\left(z_{12}z_{13} + z_{24}z_{34}\right)\mu^3 + s\mu^2 + 2z_{23}\left(z_{12}z_{24} + z_{13}z_{34}\right)\mu - z_{23}^2\Big), \quad (7)$$

where

$$s = \left(z_{34}^2 - 1\right)z_{12}^2 - 2\left(z_{14}z_{23} + z_{13}z_{24}\right)z_{34}z_{12} - z_{13}^2 + z_{14}^2 z_{23}^2 + \left(z_{13}^2 - 1\right)z_{24}^2 - z_{34}^2 - 2z_{13}z_{14}z_{23}z_{24} + 1.$$

C is, thus, a quartic/biquadratic polynomial in terms of μ (cf. [23, 38]). (Clearly, the difficulty of the two-qubit separable volume problem under study here is strongly tied to the high [fourth] degree of C in μ. By setting either $z_{14} = 0$ or $z_{23} = 0$, the degree of C can be reduced to 2 (cf. [20]).) In the *complex* case, C once again assumes the form of a quartic polynomial in μ. So one encounters, in that setting, thirteen-dimensional integration problems rather than fifteen-dimensional ones.

III. ANALYSES

So, the problem of determining the separable volumes can be seen to hinge on (in the real case), a *seven*-fold integration involving the six (independent) z_{ij}'s and μ. However, such requisite integrations, allowing μ to vary (or even holding μ constant at various values, thus, reducing to six-fold integrations), did not appear to be exactly/symbolically performable (using version 5.2 of Mathematica).

A. Estimation of the univariate functions $f_{real}(\mu)$ and $f_{complex}(\mu)$

Thus, to make further progress, it seemed necessary, at this stage, to employ numerical methods (not excluding the possibility that exact solutions might, at some point, be revealed).

We proceeded along two parallel courses, one for the 9-dimensional real two-qubit case and the other for the 15-dimensional complex case. We sought those functions $f_{real}(\mu)$ and $f_{complex}(\mu)$ that would result from imposing the conditions that the expressions A, B and C (as well as a principal 3×3 minor of ρ), along with their complex counterpart expressions, be *simultaneously* nonnegative. (The satisfaction of these joint conditions ensures that we

are dealing precisely with *separable* 4×4 density matrices.) It was evident that the relation $f(\mu) = f(\frac{1}{\mu})$ must hold, so we only studied the range $\mu \in [0, 1]$. Dividing this unit interval into 2,000 equal nonoverlapping subintervals of length $\frac{1}{2000}$ each, we sought to estimate the $f(\mu)$'s at the 2,001 end points of these subintervals.

This required (μ being fixed at these end points) numerical integrations in 6 and 12 dimensions. For this purpose, we utilized the Tezuka-Faure (TF) quasi-Monte Carlo procedure [39, 40], we had extensively used in our earlier studies of separability probabilities [7, 8]. For each of the 2,001 discrete, equally-spaced values of μ we employed the same set of 37,000,000 Tezuka-Faure six-dimensional points in the real case and, similarly, the same set of 25,000,000 twelve-dimensional points in the complex case. (The Tezuka-Faure points are defined over unit hypercubes $[0, 1]^n$, so in our computations, we transform the Bloore variables accordingly. We plan to continue to add such points to our [real and complex] analyses.)

In Figs. 1 and 2 we show the results of this procedure. There were some slight deviations from monotonicity [41] (presumably due to limited sample sizes) in the vicinity of $\mu = 1$ for both functions.

In the real case, our *estimate* of *known* Hilbert-Schmidt volume of (separable *plus* non-separable) states [2], $\frac{\pi^4}{60480} \approx 0.0016106$ was larger by only a factor 1.00006. So, we would expect our companion estimates of $f_{real}(\mu)$, at each of the 2,001 sampled points, to be roughly equally precise. (Let us note that $f_{real}(0) = f_{complex}(0) = 0$.) In the complex case, our estimate of the known 15-dimensional volume, $\frac{\pi^6}{851350500} \approx 1.12925 \cdot 10^{-6}$ was smaller only by a factor of 0.99965. (As instances of specific values, based on independent analyses using still larger numbers of TF-points, we obtained estimates of $f_{real}(1) = \frac{73430796}{640625} \approx 114.62368$, $f_{real}(\frac{1}{2}) = \frac{47475904}{640625} \approx 74.108728$, $f_{real}(\sigma_{Au}) = \frac{56575096}{640625} \approx 88.312344$, $f_{complex}(1) = 387.33307366$, $f_{complex}(\frac{1}{2}) = 180.6046580$, $f_{complex}(\sigma_{Au}) = 251.157815860$, where $\sigma_{Au} = \frac{\sqrt{5}-1}{2}$ denotes the golden ratio [17]. *Exact* characterizations of $f_{real}(\mu)$ and $f_{complex}(\mu)$ would, of course, be of great interest, in particular, for the possibility that they might yield exact volume results.)

To estimate the desired separable volumes (V_{sep}) themselves, one must perform the calculations,

$$V_{sep/real} = 2 \int_0^1 jac_{real} f_{real}(\mu) d\mu \tag{8}$$

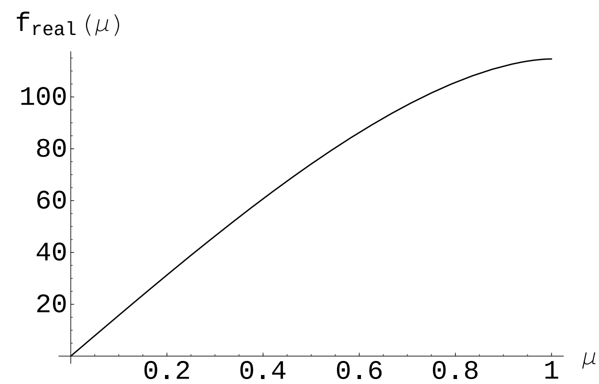

FIG. 1: Estimation of $f_{real}(\mu)$ based on the third-order interpolation of 2,001 points (μ), the value at each such point being based on *six*-fold numerical integrations employing the same (for each μ) set of thirty-seven million Tezuka-Faure points

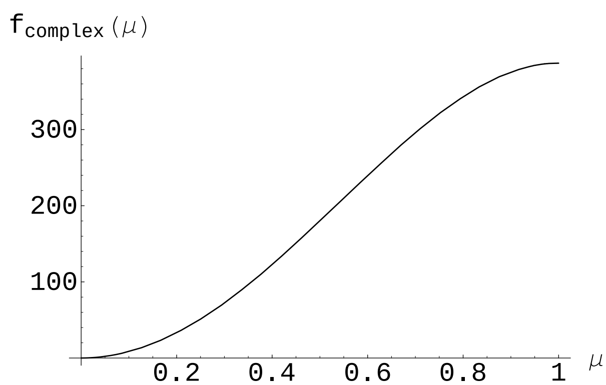

FIG. 2: Estimation of $f_{complex}(\mu)$ based on the third-order interpolation of 2,001 points (μ), the value at each such point being based on *twelve*-fold numerical integrations employing the same (for each μ) set of twenty-five million Tezuka-Faure points

and

$$V_{sep/complex} = 2 \int_0^1 jac_{complex} f_{complex}(\mu) d\mu. \tag{9}$$

B. Jacobians for the transformations

Now (Fig. 3),

$$jac_{real}(\mu) = \frac{\mu^4 \left(12 \left(\left(\mu^2+2\right)\left(\mu^4+14\mu^2+8\right)\mu^2+1\right)\log(\mu) - 5\left(5\mu^8+32\mu^6-32\mu^2-5\right)\right)}{1890\left(\mu^2-1\right)^9} \tag{10}$$

and (Fig. 4)

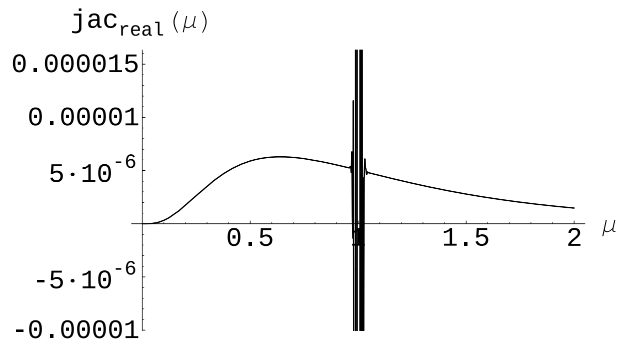

FIG. 3: Plot of the jacobian function $jac_{real}(\mu)$, given by (10)

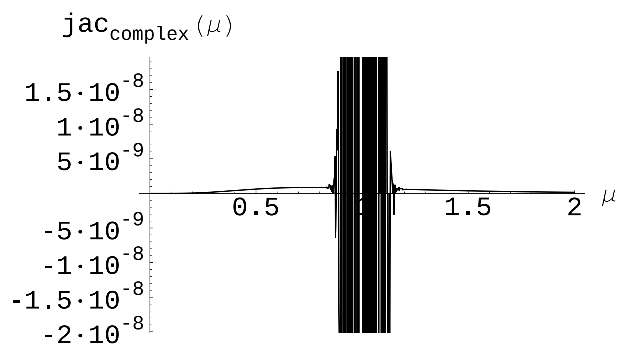

FIG. 4: Plot of the jacobian function $jac_{complex}(\mu)$, given by (11)

$$jac_{complex}(\mu) = -\frac{\mu^7}{1801800\left(\mu^2-1\right)^{15}}V, \tag{11}$$

where

$$V = 363\mu^{14} + 9947\mu^{12} + 48363\mu^{10} + 42875\mu^8 - 42875\mu^6 - 48363\mu^4 - 9947\mu^2 - 363$$

$$-140\left(\mu^{14} + 49\mu^{12} + 441\mu^{10} + 1225\mu^8 + 1225\mu^6 + 441\mu^4 + 49\mu^2 + 1\right)\log(\mu).$$

We have that

$$\int_0^1 jac_{real}(\mu)d\mu = \frac{\pi^2}{2293760} \approx 4.30281\cdot 10^{-6}, \tag{12}$$

and

$$\int_0^1 jac_{complex}(\mu)d\mu = \frac{1}{2018016000} \approx 4.95536\cdot 10^{-10}. \tag{13}$$

(The smallest value of μ for which $jac_{real}(\mu) = 0$ that we were able to find was 0.9685588023, while for $jac_{complex}(\mu)$ we found 0.8395384257.)

We obtained the jacobian functions $jac_{real}(\mu)$ and $jac_{complex}(\mu)$, given in (10) and (11), by transformations of, say, ρ_{33} to the μ variable (and subsequent two-fold exact integrations over ρ_{11} and ρ_{22}) of the original (three-dimensional) jacobians, involving the diagonal entries, for the Bloore parameterizations. These original jacobians were of the form $(\Pi_{i=1}^{4}\rho_{ii})^k$ with $k = \frac{3}{2}$ in the real case, and $k = 3$, in the complex case. (Of course, by the unit trace condition, we must have $\rho_{44} = 1 - \rho_{11} - \rho_{22} - \rho_{33}$.)

The direct high-accuracy computation of the desired separable volume integrals (8) and (9) proves challenging due to the highly oscillatory nature of $jac_{real}(\mu)$ and $jac_{complex}(\mu)$ (given by (10) and (11)) in the vicinity of $\mu = 1$, as indicated in Figs. 3 and 4. (It might be appropriate to sample more points in the vicinity of $\mu = 1$ than in other less problematical regions. We have also attempted — without significant success so far — to evaluate these integrals using repeated integration by parts [42], since the two jacobians in question admit repeated exact integrations.)

C. Volume integrals over $\mu \in [0, .95]$

Replacing the upper integration limit of 1 in the integral (8) by .95, we obtained — using high precision arithmetic — a result of 0.0006707668 and consequent *lower* bound on the probability of separability of the real two-qubit systems of 0.41647013. (The direct use of upper integration limits greater than .95 appeared to lead to unstable results.) For similar reasons, replacing the upper integration limit of 1 in the integral (9) also by .95, we obtained a result of $2.327058044 \cdot 10^{-7}$ and consequent lower bound on the probability of separability of the complex two-qubit systems of 0.2060707612.

D. Volume integrals over $\mu \in [.95, 1]$

In the immediately preceding analysis, we used upper limits of .95 rather than 1 in the integrals (8) and (9). To estimate the integrals in the remaining range $[.95, 1]$, we replaced the jacobian functions $jac_{real}(\mu)$ and $jac_{complex}(\mu)$, given in (10) and (11), by their 100-degree power series expansions about $\mu = 1$. (When plotted over [.95,1], both these replacement functions gave the appearances of simple downward-sloping *lines* (Figs. 5 and 6).)

Proceeding in such a manner, again using high-precision arithmetic and summing the

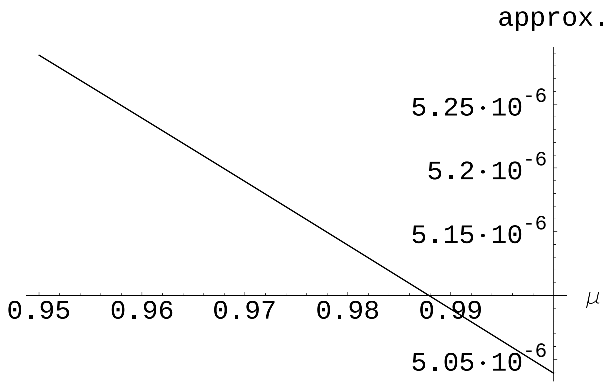

FIG. 5: 100-degree power series approximation to $jac_{real}(\mu)$ about $\mu = 1$

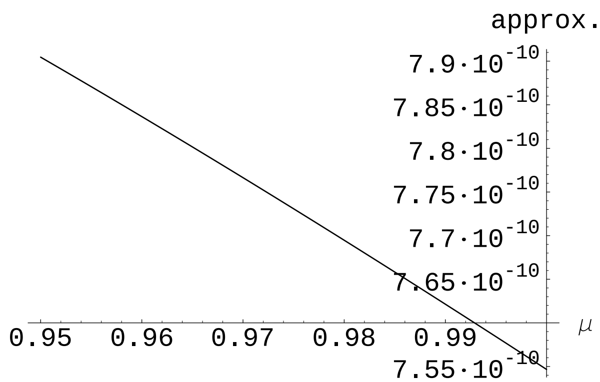

FIG. 6: 100-degree power series approximation to $jac_{complex}(\mu)$ about $\mu = 1$

results over the two sets of intervals, we arrived at our final (subject to the availability of additional Tezuka-Faure points) estimates $V_{sep/real} \approx 0.0007298112$ and $V_{sep/complex} \approx 2.625622678 \cdot 10^{-7}$. Then, we have $prob_{sep/real} \approx 0.45313001$ and $prob_{complex/real} \approx 0.23250991$. (When we compared these several results, based on interpolation — to estimate $f_{real}(\mu)$ and $f_{complex}(\mu)$ (Figs. 1 and 2) — using *third*-degree polynomials with those using *sixth*-degree polynomials, we obtained essentially the same set of results.)

As noted in the introductory section, we had previously hypothesized that $V_{sep/complex} = (5\sqrt{3})^{-7} \approx 2.73707 \cdot 10^{-7}$ [8, eq. (41)] and $prob_{sep/complex} = \frac{2^2 \cdot 3 \cdot 7^2 \cdot 11 \cdot 13\sqrt{3}}{5^4 \pi^6} \approx 0.242379$ [8, eq. (43), but misprinted as 5^3 not 5^4 there]. The analysis there was based on a considerably larger number — 400,000,000 — of Tezuka-Faure points than here. But each point there was employed only *once* for the Peres-Horodecki separability test, while each point here is used in 2,000 such tests (with μ ranging over [0,1]). (Relatedly, we had initially suspected that if we started checking the Peres-Horodecki criterion for successively larger values of μ,

holding the set of z_{ij}'s given by a Tezuka-Faure point *fixed*, then if we reached one value for which separability held, then *all* higher values of μ would also yield separability. But this turned out not to be invariably the case. So, it appeared that we needed to check the criterion 2,000 times for each point.)

IV. CONCLUDING REMARKS

In our earlier study [20], we had also employed the Bloore parameterization of the two-qubit (and qubit-qutrit) systems to study the Hilbert-Schmidt separability probabilities of specialized systems of less than full dimensionality. We also reported an effort to determine a certain *three*-dimensional function (in contrast to the *one*-dimensional functions $f_{real}(\mu)$ and $f_{complex}(\mu)$ above, but for somewhat a similar purpose) over the simplex of eigenvalues that would facilitate the calculation of the 15-dimensional volume of the two-qubit systems in terms of (monotone) metrics — such as the Bures, Kubo-Mori, Wigner-Yanase,... — other than the (non-monotone [43]) Hilbert-Schmidt one considered here. (The Bloore parameterization [19], used above, did not seem immediately useful in this monotone metric context, since the *eigenvalues* of ρ are not explicitly expressed (cf. [44]). Therefore, we had recourse in [20] to the Euler-angle parameterization of Tilma, Byrd and Sudarshan [23].)

Let us direct the reader to some papers of R. Kellerhals concerned, among other items, with the volumes of *hyperbolic* polyhedra [45, 46] (cf. [47]). In this line of work, the dilogarithm and, more generally, the polylogarithm functions play important roles. There have been some indicators in our investigations above (in particular, in integrations of the jacobians) that these functions also may be of relevance in our context.

The extension to qubit-*qutrit* pairs (and even higher-dimensional compositie systems) of the univariate-function-strategy we have pursued above, for the case of qubit-qubit pairs, seems problematical, although we have not yet examined the matter in great detail. In the qubit-qubit case, the analysis is facilitated by the fact that it is sufficient that the determinant of the partial transpose be nonnegative for the Peres-Horodecki separability criterion to hold [36, Thm. 5]. More requirements than this single one are needed in the qubit-qutrit scenario — even though the criterion on the nonnegativity of the partial transpose is still both necessary and sufficient for 6×6 density matrices. (In addition to the determinant, the leading minors and/or the individual eigenvalues of the partial transpose

of the 6×6 density matrix would need to be tested for nonnegativity, as well. Also the qubit-qutrit analogue of the ratio (μ) of diagonal entries would have to be defined, if even possible.)

Acknowledgments

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